Near-Optimal Coding for Many-Access Channels

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Joint work with Kuan Hsieh (Cambridge), Cynthia Rush (Columbia)

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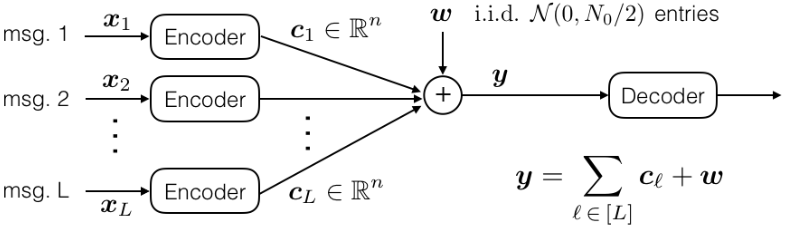
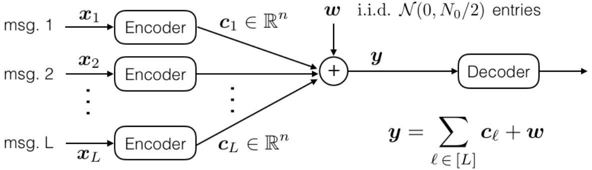
Gaussian multiple-access channel



Modern networks often have

- Very large number of users
- Small data payload for each user

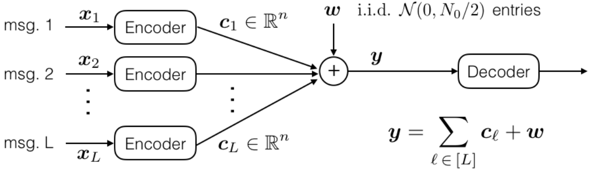
Many-user setting



- Achievable user density $\mu = L/n$
- Fixed user payload $\log M$ bits/user
- Energy-per-bit constraint $\|\boldsymbol{c}_i\|^2 \leq E := E_b \log M, \; i \in [L]$
- Per-user probability of error (PUPE) $\frac{1}{L}\sum_i \mathbb{P}(\hat{\boldsymbol{x}}_i \neq \boldsymbol{x}_i)$

[Chen, Chen, Guo, '17], [Ravi, Koch '19] [Polyanskiy '17], [Zadik, Polyanskiy, Thrampoulidis '19] , [Polyanskiy, Kowshik '20]

Many-user setting



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Linear scaling regime

$L, n \to \infty$ with $\mu = L/n$ fixed, E_b and M do not scale with n

What is minimum E_b/N_0 required for a given μ and target PUPE, e.g. 10^{-3} ?

[Chen, Chen, Guo, '17], [Ravi, Koch '19] [Polyanskiy '17], [Zadik, Polyanskiy, Thrampoulidis '19] , [Polyanskiy, Kowshik '20]

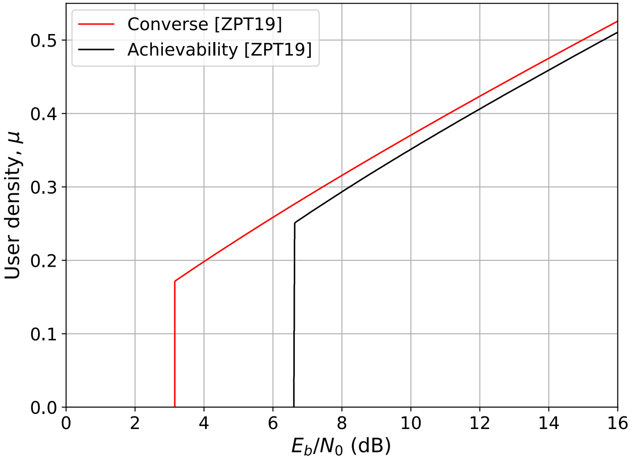
Previous work

What can be achieved with random Gaussian codebooks and (infeasible) maximum-likelihood decoding?

This talk

What can be achieved with random linear coding and efficient Approximate Message Passing (AMP) decoding?

Bounds

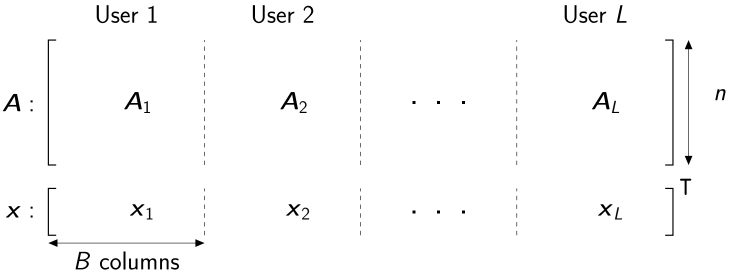


User payload = 8 bits

For each E_b/N_0 value, find max. μ that achieves PUPE $\leq 10^{-3}$.

[Zadik, Polyanskiy, Thrampoulidis '19], [Polyanskiy, Kowshik '20]

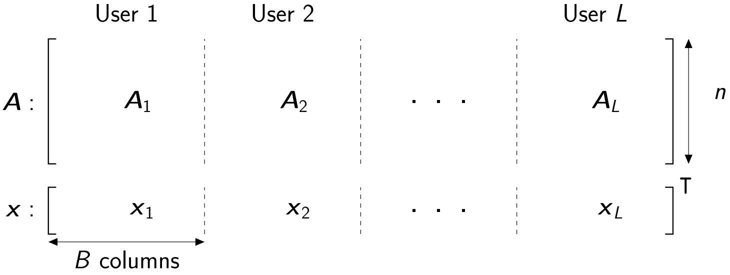
Random linear coding



For each user i, codeword $\boldsymbol{c}_i = \boldsymbol{A}_i \boldsymbol{x}_i$

- Random matrices: $\boldsymbol{A}_i \in \mathbb{R}^{n \times B}$
- User i's message encoded in $\boldsymbol{x}_i \in \mathbb{R}^B \sim P_{\boldsymbol{X}}$

$$\boldsymbol{y} = \sum_i \boldsymbol{A}_i \boldsymbol{x}_i + \boldsymbol{w} = \boldsymbol{A}\boldsymbol{x} + \boldsymbol{w}$$



$$y = \sum_i A_i x_i + w = Ax + w$$

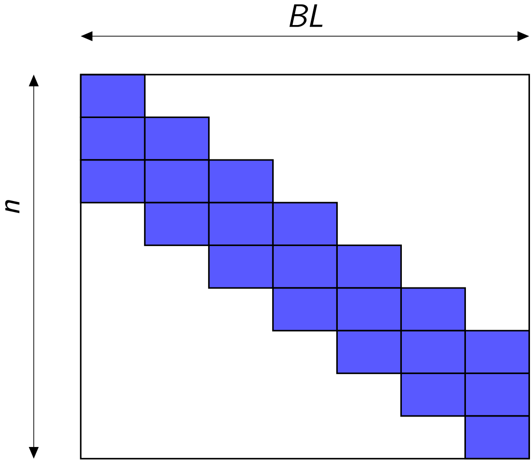
Examples with IID Gaussian $\boldsymbol{A}$

- Random codebooks: $B = M$, and each $\boldsymbol{x}_i$ has a single nonzero value $= \sqrt{E}$
- Random CDMA: $B = 1$ and $\boldsymbol{x}_i$ drawn from M-ary constellation

  ⋮

We will also use *spatially coupled* $\boldsymbol{A}$

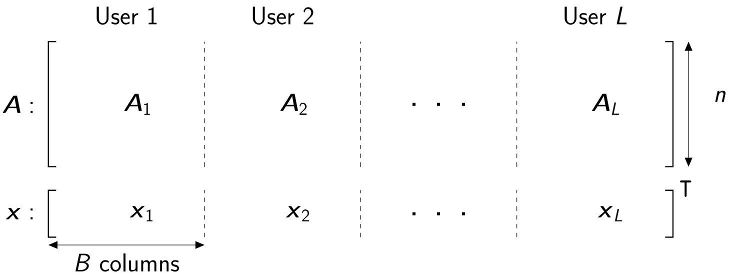
Spatially coupled matrix



Combined codebook matrix A

Gaussian entries on band-diagonal, remaining entries zero

IID Gaussian matrix

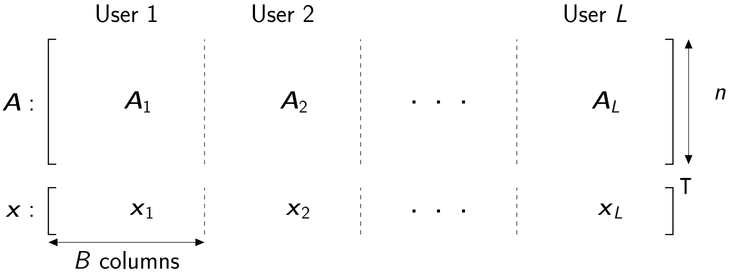


$$A_{jk} \sim_{iid} \mathcal{N}(0, 1/n), \qquad \boldsymbol{x}_i \sim_{iid} P_{\boldsymbol{X}}$$

Decoding task: Recover $\boldsymbol{x}_1, \ldots, \boldsymbol{x}_L$ from

$$\boldsymbol{y} = \sum_i \boldsymbol{A}_i \boldsymbol{x}_i + \boldsymbol{w} = \boldsymbol{A}\boldsymbol{x} + \boldsymbol{w}$$

Approximate Message Passing



AMP decoder to recover $\boldsymbol{x} = [\boldsymbol{x}_1, \ldots, \boldsymbol{x}_L]$

Initialize with $\boldsymbol{x}^0 = \mathbf{0}$, and for $t \geq 0$:

$$\text{Modified residual} \; : \quad \boldsymbol{z}^t = \boldsymbol{y} - \boldsymbol{A}\boldsymbol{x}^t + \upsilon^t \boldsymbol{z}^{t-1}$$

[Donoho, Maleki, Montanari '09], [Bayati, Montanari '11], [Barbier, Krzakala '15], [Rush, V '15], . . .

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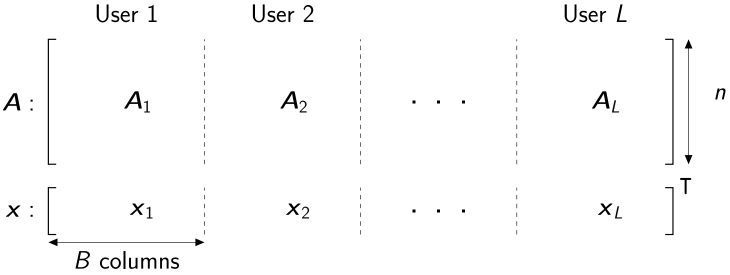

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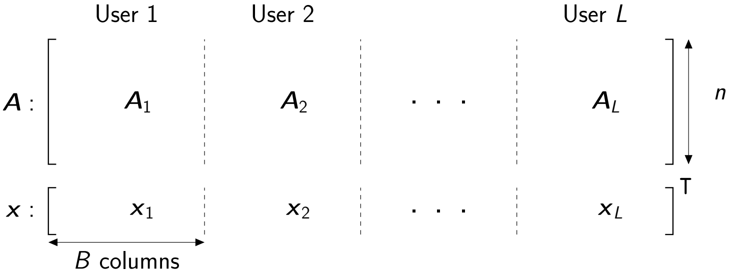

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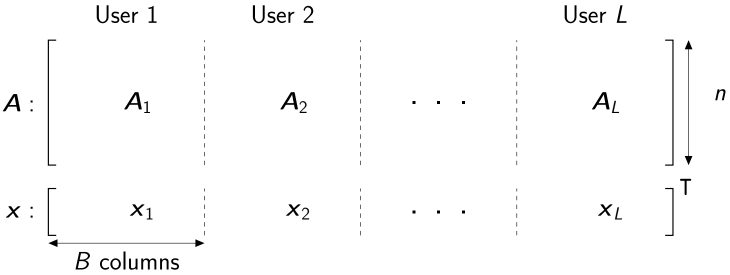
Key distributional property

Empirical distribution of $(\boldsymbol{s}^t - \boldsymbol{x}) \xrightarrow{W_2} \mathcal{N}(0, \tau^t)$

Approximate Message Passing



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$$\text{New estimate for user } i : \quad \boldsymbol{x}_i^{t+1} = \mathbb{E}[\, \boldsymbol{X} \mid \boldsymbol{X} + \sqrt{\tau^t}\boldsymbol{G} = \boldsymbol{s}_i^t] \quad \in \mathbb{R}^B$$

$$\boldsymbol{X} \sim P_{\boldsymbol{X}}, \;\; \boldsymbol{G} \sim \mathcal{N}(\mathbf{0}, \boldsymbol{I}_B)$$

State evolution

Modified residual: $\boldsymbol{z}^t = \boldsymbol{y} - \boldsymbol{A}\boldsymbol{x}^t + \upsilon^t \boldsymbol{z}^{t-1}$

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For $t \geq 0$, the effective noise variance is

$$\tau^{t+1} = \frac{N_0}{2} + \mu \cdot \text{mmse}\left(\frac{1}{\tau^t}\right)$$

where

$$\text{mmse}(1/\tau^t) = \mathbb{E}\left\{\|\boldsymbol{X} - \mathbb{E}[\boldsymbol{X} \mid \boldsymbol{X} + \sqrt{\tau^t}\boldsymbol{G}]\|^2\right\}, \;\; \boldsymbol{X} \sim P_{\boldsymbol{X}}, \;\; \boldsymbol{G} \sim \mathcal{N}(\boldsymbol{0}, \boldsymbol{I}_B)$$

Initialize with $\tau^0 = \frac{N_0}{2} + \mu E$

Termination

After a large number of iterations t:

$$\text{Effective observation}: \quad \boldsymbol{s}^t = \boldsymbol{x}^t + \boldsymbol{A}^\mathsf{T} \boldsymbol{z}^t$$

$$\text{Estimate for user } i: \quad \boldsymbol{x}_i^{t+1} = \mathbb{E}[\boldsymbol{X} \mid \boldsymbol{X} + \sqrt{\tau^t}\boldsymbol{G} = \boldsymbol{s}_i^t]$$

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Hard-decision estimate:

$$\hat{\boldsymbol{x}}_i^{t+1} = \arg\max_{\boldsymbol{x}' \in \mathcal{X}} \mathbb{P}\left(\boldsymbol{X} = \boldsymbol{x}' \mid \boldsymbol{X} + \sqrt{\tau^t}\boldsymbol{G} = \boldsymbol{s}_i^t\right), \qquad i \in [L]$$

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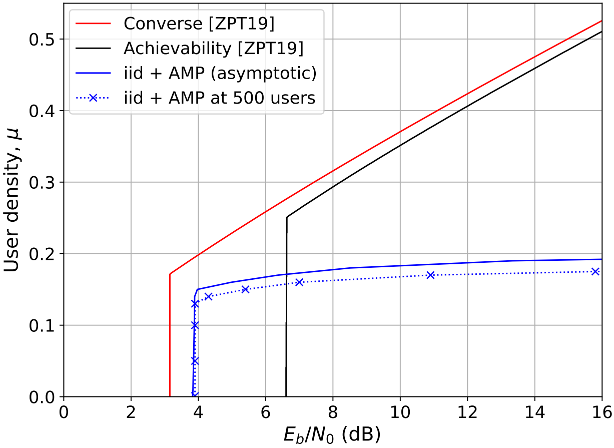
What happens as $t \to \infty$ to

$$\tau^t = \frac{N_0}{2} + \mu \cdot \text{mmse}(1/\tau^{t-1})$$

This determines the user error rate:

$$\frac{1}{L}\sum_{i=1}^{L} \mathbb{1}\{\hat{\boldsymbol{x}}_i^{t+1} \neq \boldsymbol{x}_i\}$$



User payload $= 8$ bits

For each μ, we find minimum E_b/N_0 that achieves PUPE $\leq 10^{-3}$

Theoretical curve is derived from the single-user effective channel

Single-user channel

$$\boldsymbol{S}_\tau = \boldsymbol{X} + \sqrt{\tau}\boldsymbol{G}, \qquad \boldsymbol{X} \sim P_{\boldsymbol{X}},\ \boldsymbol{G} \sim \mathcal{N}(\boldsymbol{0}, \boldsymbol{I}_B)$$

MAP estimator: $\hat{\boldsymbol{x}}^{\mathrm{MAP}}(\boldsymbol{S}_\tau) = \arg\max_{\boldsymbol{x}' \in \mathcal{X}} \ \mathbb{P}(\boldsymbol{X} = \boldsymbol{x}' \mid \boldsymbol{S}_\tau)$

Prob. of error: $P_e(\tau) = \mathbb{P}(\hat{\boldsymbol{x}}^{\mathrm{MAP}}(\boldsymbol{S}_\tau) \neq \boldsymbol{X})$

Example: Random Gaussian codebooks

$$\hat{\boldsymbol{x}}_j^{\mathrm{MAP}}(\boldsymbol{s}) = \begin{cases} \sqrt{E} & \text{if } s_j > s_k \text{ for all } k \in [B]\backslash j, \\ 0 & \text{otherwise} \end{cases}$$

$$P_e(\tau) = 1 - \mathbb{E}\left[\Phi(\sqrt{E/\tau} + G)^{B-1}\right]$$

Theorem

Consider iid Gaussian $\boldsymbol{A}$ and message vectors $\boldsymbol{x}_i \sim_{iid} P_{\boldsymbol{X}}$. Then, the asymptotic user error rate of the AMP decoder is

$$\lim_{t\to\infty} \lim_{L\to\infty} \frac{1}{L} \sum_{\ell=1}^{L} \mathbb{1}\left\{\hat{\boldsymbol{x}}_\ell^t \neq \boldsymbol{x}_\ell\right\} \stackrel{\text{a.s.}}{=} P_e(\tau^{\mathrm{FP}})$$

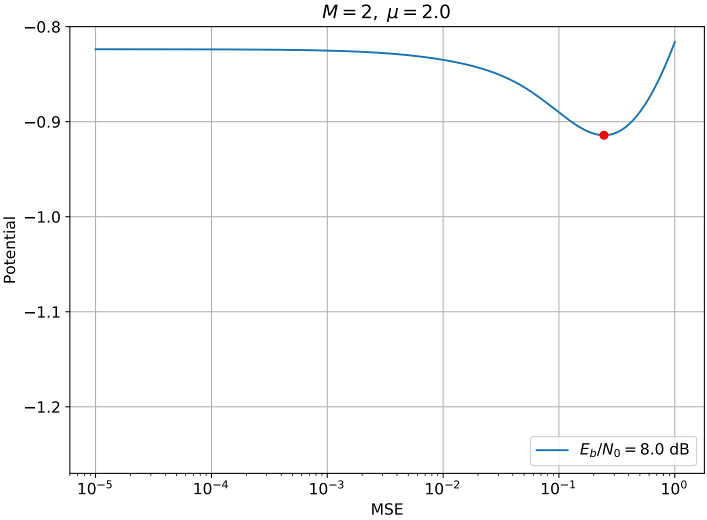
where the inner limit is taken with $L/n = \mu$.

τ^{FP} is the **largest stationary point** of the **potential function**:

$$\mathcal{F}(\tau) = I(\boldsymbol{X}; \boldsymbol{S}_\tau) + \frac{1}{2\mu}\left[\ln\left(\frac{\tau}{N_0/2}\right) - \left(1 - \frac{N_0/2}{\tau}\right)\right]$$

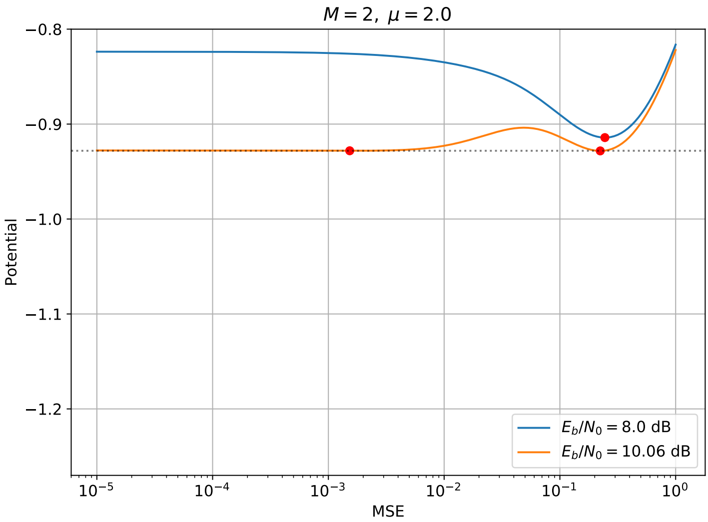
where $\tau \in \left[\frac{N_0}{2}, \frac{N_0}{2} + \mu E\right]$.

Potential function



x-axis is $\left(\tau - \frac{N_0}{2}\right)\frac{1}{\mu}$

Potential function



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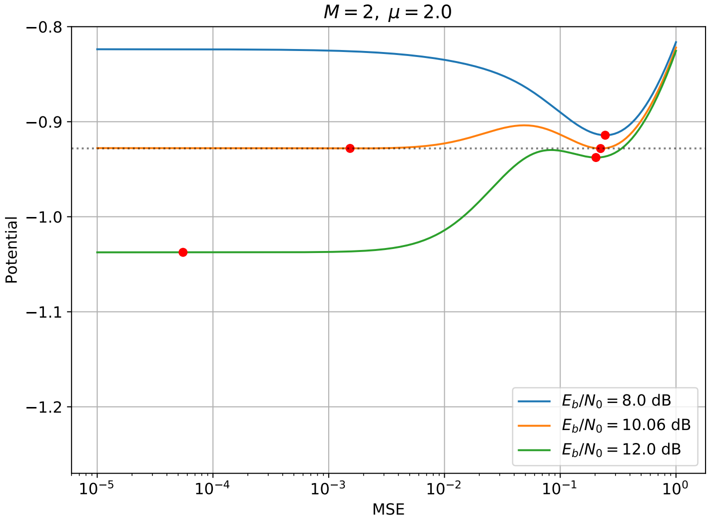

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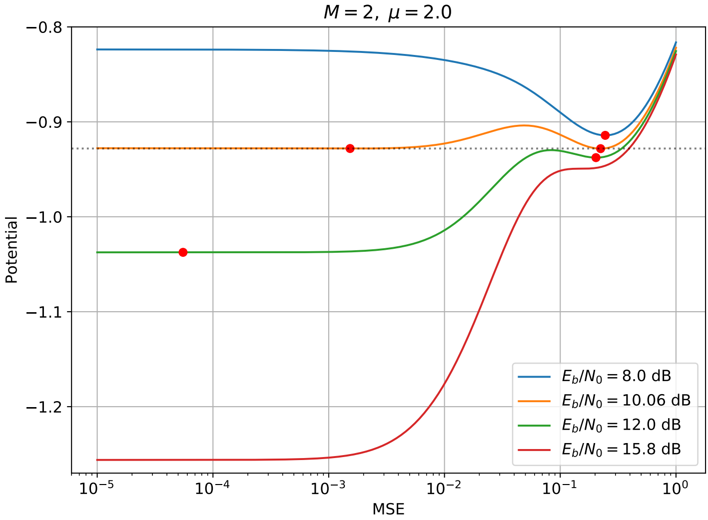

Can we achieve $P_e(\tau^*)$, corresponding to the *global minimum*?

MMSE estimator

Consider $\boldsymbol{y} = \boldsymbol{A}\boldsymbol{x} + \boldsymbol{w}$ with $\boldsymbol{x} \sim_{\text{iid}} P_{\boldsymbol{X}}$ and $\boldsymbol{w} \sim_{\text{iid}} \mathcal{N}(0, \frac{N_0}{2})$

Decoder that minimizes the MSE: $\hat{\boldsymbol{x}}^{\text{mmse}} = \mathbb{E}[\boldsymbol{x} \mid \boldsymbol{y}, \boldsymbol{A}]$

Limiting MMSE: $\lim_{L\to\infty} \frac{1}{L}\mathbb{E}\left\{\|\boldsymbol{x} - \hat{\boldsymbol{x}}^{\text{mmse}}\|^2\right\}$

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Limiting MMSE: $\lim_{L\to\infty} \frac{1}{L}\mathbb{E}\left\{\|\boldsymbol{x} - \hat{\boldsymbol{x}}^{\text{mmse}}\|^2\right\}$

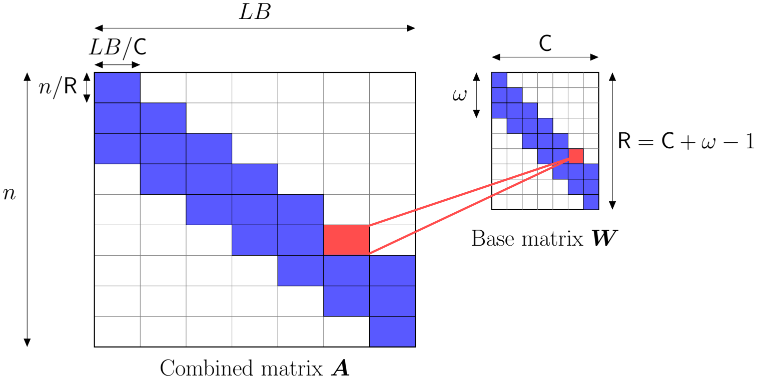
From [Reeves, Pfister '16], [Barbier et al. '17] :

$$\frac{1}{L}\mathbb{E}\left\{\|\boldsymbol{x} - \hat{\boldsymbol{x}}^{\text{mmse}}\|^2\right\} \;\to\; \left(\tau^* - \frac{N_0}{2}\right)\frac{1}{\mu}$$

where τ^* is the *global minimum* of the potential function

$$\mathcal{F}(\tau) = I(\boldsymbol{X}; \boldsymbol{S}_\tau) + \frac{1}{2\mu}\left[\ln\left(\frac{\tau}{N_0/2}\right) - \left(1 - \frac{N_0/2}{\tau}\right)\right]$$

Spatially coupled Gaussian matrix



$$A_{jk} \sim \mathcal{N}(0, W_{\mathsf{rc}}) \text{ for } j \in \text{block r}, \; k \in \text{block c}$$

$\boldsymbol{x} = [\boldsymbol{x}_1, \boldsymbol{x}_2, \dots, \boldsymbol{x}_L]$ has same form as before: $\boldsymbol{x}_i \in \mathbb{R}^B \sim_{iid} P_{\boldsymbol{X}}$

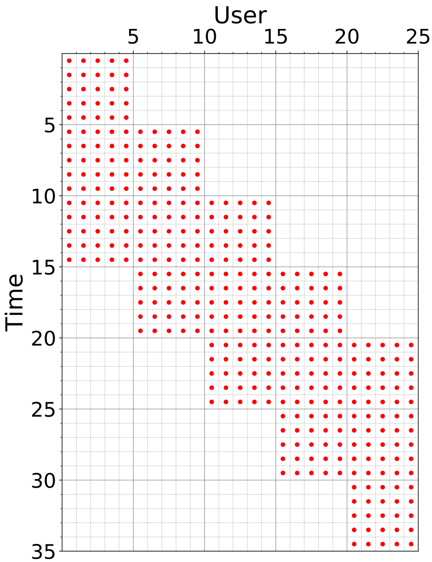
[Donoho, Javanmard, Montanari '13] [Barbier and Krzakala '17] [Liang, Ma and Ping '17] [Hsieh, Rush, V '21] . . .

Example:

$L = 25$ users

$n = 35$ channel uses

$(\omega = 3, \mathrm{C} = 5)$ base matrix



Spatial coupling induces **block-wise time-division with overlap**

AMP Decoder

Initialize with $\boldsymbol{x}^0 = \mathbf{0}$, and for $t \geq 0$:

$$\text{Modified residual:} \quad \boldsymbol{z}^t = \boldsymbol{y} - \boldsymbol{A}\boldsymbol{x}^t + \tilde{\boldsymbol{\upsilon}}^t \odot \boldsymbol{z}^{t-1}$$

$$\text{Effective observation:} \quad \boldsymbol{s}^t = \boldsymbol{x}^t + (\tilde{\boldsymbol{S}}^t \odot \boldsymbol{A})^\top \boldsymbol{z}^t$$

$$\text{New estimate for user } i: \quad \boldsymbol{x}_i^{t+1} = \mathbb{E}[\,\boldsymbol{X} \mid \boldsymbol{X} + \sqrt{\tau_i^t}\boldsymbol{G} = \boldsymbol{s}_i^t]$$

$$\boldsymbol{X} \sim P_{\boldsymbol{X}}, \;\; \boldsymbol{G} \sim \mathcal{N}(\mathbf{0}, \boldsymbol{I}_B)$$

$\boldsymbol{\tau}^t = (\tau_1^t, \ldots, \tau_L^t)$ specifies effective noise variance for each user

$\tilde{\boldsymbol{\upsilon}}^t$, $\tilde{\boldsymbol{S}}^t$, $\boldsymbol{\tau}^t$ determined via state evolution recursion

Theorem (Threshold Saturation)

Consider spatially coupled Gaussian $\boldsymbol{A}$, message vectors $\boldsymbol{x}_i \sim_{iid} P_{\boldsymbol{X}}$.

For any $\delta > 0$, sufficiently large ω and sufficiently small $\frac{\omega}{\mathsf{C}}$ the asymptotic user error rate of the AMP decoder satisfies

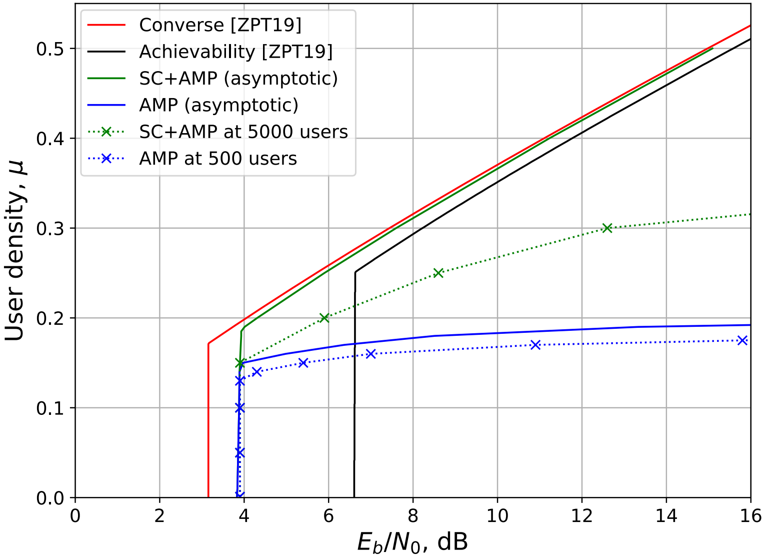
$$\lim_{t\to\infty} \lim_{L\to\infty} \frac{1}{L} \sum_{\ell=1}^{L} \mathbb{1}\left\{\hat{\boldsymbol{x}}_\ell^t \neq \boldsymbol{x}_\ell\right\} \leq P_e(\tau^* + \delta) \quad \text{a. s.}$$

where the inner limit is taken with $L/n = \mu$.

Here τ^* is the **global minimum** of the potential function:

$$\mathcal{F}(\tau) = I(\boldsymbol{X}; \boldsymbol{S}_\tau) + \frac{1}{2\mu}\left[\ln\left(\frac{\tau}{N_0/2}\right) - \left(1 - \frac{N_0/2}{\tau}\right)\right]$$

where $\tau \in \left[\frac{N_0}{2}, \frac{N_0}{2} + \mu E\right]$.



User payload = 8 bits

For each μ, we find minimum E_b/N_0 that achieves PUPE $\leq 10^{-3}$

Spectral efficiency

For fixed E_b/N_0 and target PUPE ε, achievable user density $\to 0$ for iid Gaussian codebooks as user payload $\log M$ increases

Interesting regime for large payloads:

- Constant **spectral efficiency**:

$$S := \mu \log M = \frac{L \log M}{n}$$

- We will consider the limit $L, n, M \to \infty$ with S constant

Theorem

1) Let

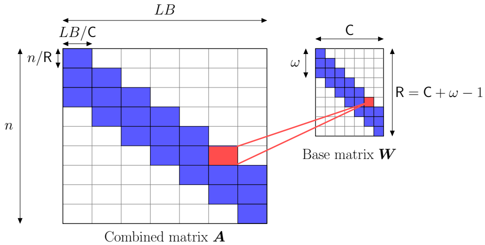
$$S_{\text{BP}} = \frac{1}{2}\left(\frac{1}{\ln 2} - \frac{1}{E_b/N_0}\right).$$

Consider random coding with **i.i.d. Gaussian codebooks**. The user error rate of the AMP decoder for any $t \geq 1$ satisfies:

$$\lim_{L,M,n\to\infty} \frac{1}{L}\sum_{\ell=1}^{L} \mathbb{1}\left\{\hat{\boldsymbol{x}}_\ell^t \neq \boldsymbol{x}_\ell\right\} \quad = \begin{cases} 0 & \text{if } S < S_{\text{BP}}, \\ 1 & \text{otherwise,} \end{cases}$$

S_{BP} is called the **belief propagation threshold**

Theorem



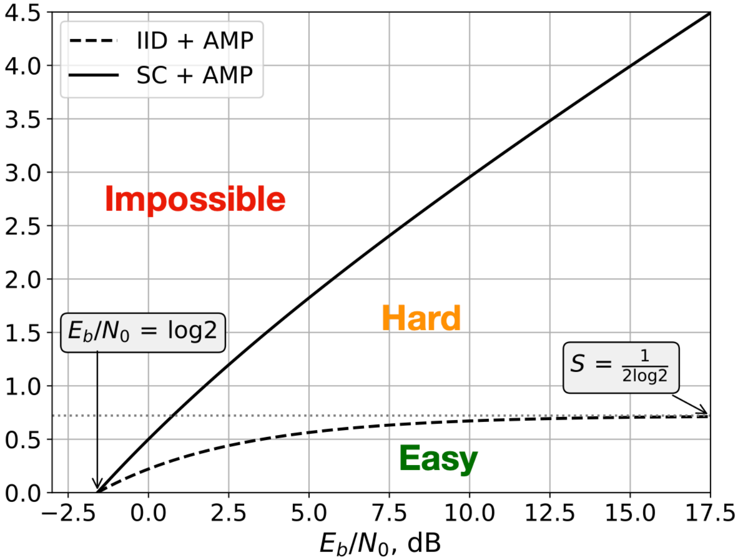
2) Let S_{opt} be the solution to

$$S_{\text{opt}} = \frac{1}{2}\log\left(1 + \frac{E_b S_{\text{opt}}}{N_0/2}\right).$$

Consider random coding with a **spatially coupled $\boldsymbol{A}$**. For $S_{BP} \leq S < S_{\text{opt}}$, the error rate of the AMP decoder satisfies:

$$\lim_{L,M,n\to\infty} \frac{1}{L}\sum_{\ell=1}^{L} \mathbb{1}\left\{\hat{\boldsymbol{x}}_\ell^t \neq \boldsymbol{x}_\ell\right\} = 0$$

for t and (ω, C) sufficiently large and ω/C sufficiently small


IID + AMP
SC + AMP
Impossible
Hard
Easy
$E_b/N_0 = \log 2$
$S = \frac{1}{2\log 2}$
4.5
4.0
3.5
3.0
2.5
2.0
1.5
1.0
0.5
0.0
−2.5
0.0
2.5
5.0
7.5
10.0
12.5
15.0
17.5
E_b/N_0, dB

Performance of optimal decoding with iid/uncoupled design can be matched by *spatially coupled design with message passing decoding*

Future directions

Efficient schemes for larger user payloads, e.g., $\log M = 100$ bits

- Finding stationary points of potential function infeasible for very large M.
- Can get bounds by analyzing suboptimal AMP decoder which has a simpler potential function [Kowshik '22]
- Decoder implementation still challenging as size of codebook matrix grows

Performance of optimal decoding with iid/uncoupled design can be matched by *spatially coupled design with message passing decoding*

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Efficient schemes for larger user payloads, e.g., $\log M = 100$ bits

- Finding stationary points of potential function infeasible for very large M.
- Can get bounds by analyzing suboptimal AMP decoder which has a simpler potential function [Kowshik '22]
- Decoder implementation still challenging as size of codebook matrix grows

Spatially coupled design for random access MACs

- Can we induce spatial coupling when users begin transmissions at random times?
- In many applications, the number of users is also random

Spatially coupled design with message passing decoding can match performance of optimal decoding on uncoupled design

Spatially coupling for generalized linear models

$$\boldsymbol{y} = f(\boldsymbol{A}\boldsymbol{x}, \boldsymbol{w}), \qquad \boldsymbol{x} \sim P_X \quad \boldsymbol{w} \sim P_W$$

Examples:

Phase retrieval: $\boldsymbol{y} = |\boldsymbol{A}\boldsymbol{x}|^2 + \boldsymbol{w}$

1-bit measurements: $\boldsymbol{y} = \text{sign}(\boldsymbol{A}\boldsymbol{x})$

Formula for asymptotic MMSE by [Barbier et al, '19]

Can use spatially coupled $\boldsymbol{A}$ with AMP to approach MMSE

Decoding propagation